TIME-INDEPENDENT TEMPERATURE FIELD IN A BAND AND A LAYER UNDER MIXED BOUNDARY CONDITIONS

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The time-independent temperature field is determined for an infinite band and a layer under mixed boundary conditions.

1. Suppose that an infinite band of width d contains stationary sources of heat. On one surface we have the boundary condition of the third kind; on part of the other surface, which has a length $2l$, we have a given heat flux, and the remaining part is kept at zero temperature. We shall use dimensionless coordinates referred to the quantity l and the dimensionless width of the band $\delta = d/l$. The boundary conditions will be taken in the form:

$$\text{for } |x| < \infty, \quad y = \delta \quad \frac{\partial T}{\partial y} + hl(T - T_c) = 0; \tag{1}$$

$$\text{for } y = 0 \quad T = 0 \quad (|x| \geqslant 1), \quad \frac{\partial T}{\partial y} = q(x) \quad (|x| \leqslant 1). \tag{2}$$

It is required to determine the temperature field in the band.

The temperature will be written as the sum of two terms

$$T(x, y) = t^*(x, y) + t(x, y) \tag{3}$$

where $t^*(x, y)$ is the main temperature field in the band containing the heat sources, when the boundary condition (1) is satisfied on $y = \delta$ and the other surface is held at zero temperature. The function $t^*(x, y)$ will be assumed to be known since it can readily be found through the integral Fourier transform.

The main problem is to determine the additional field $t(x, y)$ due to imposition of the mixed boundary conditions for $y = 0$. For the additional temperature field we have the following boundary conditions:

$$\begin{gathered} \text{for } |x| < \infty, \quad y = \delta \quad \frac{\partial t}{\partial y} + hlt = 0; \\ \text{for } y = 0 \quad t = 0 \quad (|x| \geqslant 1), \quad \frac{\partial t}{\partial y} = f(x) \, (|x| \leqslant 1), \\ f(x) = q(x) - \frac{\partial t^*(x, 0)}{\partial y}. \end{gathered} \tag{4}$$

Taking the integral Fourier transform of the heat transfer equation $\Delta t = 0$, and using the boundary conditions (4), we obtain the following singular integral equation for $t'(x) = \partial t(x, 0)/\partial x$:

$$\int_{-1}^{1} t'(\xi) K\left(\frac{\xi - x}{\delta}\right) d\xi = \pi\delta f(x) \quad (|x| \leqslant 1, \ y = 0), \tag{5}$$

$$K(u) = \int_0^\infty L(\eta) \sin(\eta u) \, d\eta, \qquad L(\eta) = \frac{\eta \operatorname{th} \eta + hd}{\eta + hd \operatorname{th} \eta}. \tag{6}$$

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In particular, substituting $h = \infty$ and $h = 0$ in Eq. (6), we obtain the expression for $L(\eta)$ for the case where on the upper surface of the band we have, respectively, the boundary condition of the first and second kind:

$$1)\quad L(\eta) = \operatorname{cth} \eta; \qquad 2)\quad L(\eta) = \operatorname{th} \eta. \tag{7}$$

The kernel K(u) can be written in the form

$$K(u) = \frac{1}{u} + \sum_{k=1}^{\infty} b_k u^{2k-1}, \tag{8}$$

where

$$b_k = \frac{(-1)^{k-1}}{(2k-1)!} \int_0^{\infty} [L(\eta) - 1]\, \eta^{2k-1}\, d\eta. \tag{9}$$

These coefficients are, in general, determined by numerical integration, but when $hd > 1$ we can obtain an approximate expression for them. With this in view, we approximate the function $L(\eta)$ as follows:

$$L(\eta) = \operatorname{cth} A\eta, \quad A = \frac{1 + hd}{hd}. \tag{10}$$

The maximum relative error introduced by the use of this approximation when $hd = 1$ does not exceed 3.8%, and falls very rapidly with increasing hd. We then have from Eq. (9)

$$b_k = \frac{(-1)^{k-1} |B_{2k}|}{(2k)!} \left(\frac{\pi hd}{1 + hd} \right)^{2k}, \quad hd > 1. \tag{11}$$

When $L(\eta)$ is given by Eq. (7) or (10), Eq. (5) can be written in the form

$$\int_a^b \frac{s^{\frac{k-1}{2}}\, t'(s)}{s - v}\, ds = \frac{A\delta}{v^{\frac{3-k}{2}}}\, f^*(v). \tag{12}$$

where $a = \exp(-\pi/\delta A)$, $b = \exp(\pi/\delta A)$, $s = \exp(\pi\xi/A\delta)$, $v = \exp(\pi x/A\delta)$, $f^*(v) = f(x)$. When $k = 1$, we have the boundary condition of the first and third kinds, and $k = 2$ corresponds to the boundary condition of the second kind. At the same time, if $L(\eta)$ is given by Eq. (7), we must set $A = 1$ in Eq. (12).

Using the Cauchy type inversion formulas in Eq. (12), we obtain

$$t'(v) = -\frac{A\delta}{\pi v^{\frac{k-1}{2}} X(v)} \left\{ \frac{1}{\pi} \int_a^b \frac{X(s) f^*(s)}{s^{\frac{3-k}{2}} (s - v)}\, ds + C_0 \right\}, \tag{13}$$

where $X(v) = \sqrt{(b-v)(v-a)}$.

Integration of Eq. (13) yields the temperature $t(x)$ on the lower face of the band for $|x| < 1$. The arbitrary integration constant and the constant C_0 can be determined from the condition $t(\pm 1) = 0$.

When $L(\eta)$ is given by Eq. (6) and hd is arbitrary, Eq. (5) is solved approximately. Since the series in Eq. (8) converges absolutely for $u < 2$, all the subsequent conclusions are valid for $1 < \delta < \infty$.

Equation (5) will be solved by the asymptotic method developed in [1-3]. This is done by substituting Eq. (8) in Eq. (5) with the result

$$\int_{-1}^{1} \frac{t'(\xi)\, d\xi}{\xi - x} = \pi f(x) - \sum_{k=1}^{\infty} \frac{b_k}{\delta^{2k}} \int_{-1}^{1} t'(\xi) (\xi - x)^{2k-1}\, d\xi. \tag{14}$$

The solution of this equation will be sought in the form of a series in powers of δ^{-2}

$$t'(x) = \sum_{j=0}^{\infty} \delta^{-2j} t_j'(x). \tag{15}$$

Substituting Eq. (15) in Eq. (14), and equating the expressions in front of equal powers of δ, we obtain a set of integral equations for the functions $t_n'(x)$, the solution of which can be obtained by inverting the Cauchy integral:

TABLE 1. Maximum Temperature on the Lower Surface of the Band and the Layer as a Function of δ

δ	1,5	2	2,5	3	5	10	∞
$\frac{\delta}{T_0} t_e$	0,8556	0,9104	0,9405	0,9567	0,9835	0,9956	1
$\frac{\delta}{T_0} t_a$	0,8896	0,9209	0,9443	0,9593	0,9842	0,9959	1
$\frac{\pi\delta}{2T_0} t_c$	0,9668	0,9856	0,9923	0,9955	0,9990	0,9999	1

$$t_0'(x) = -\frac{1}{\pi\sqrt{1-x^2}}\left[\int_{-1}^{1}\frac{\sqrt{1-\xi^2}\,f(\xi)}{\xi - x}\,d\xi + C\right],$$

$$t_n'(x) = \frac{1}{\pi^2\sqrt{1-x^2}}\left[\int_{-1}^{1}\frac{\sqrt{1-\xi^2}\,d\xi}{\xi - x}\int_{-1}^{1}\sum_{k=0}^{n-1} b_{n-k}t_k'(s)(s-\xi)^{2(n-k)-1}\,ds\right], \tag{16}$$

$$n = 1, 2, \ldots$$

After integration subject to the condition $t(\pm 1) = 0$, we obtain the temperature $t(x)$ for $|x| < 1$, $y = 0$.

Consider an example. Suppose that on the upper surface of the band we have a constant temperature $T(x, \delta) = T_0$ and that for $|x| < 1$ the lower surface is thermally insulated, i.e., $q(x) = 0$. We then have $t^*(x, y) = T_0 y/\delta$ and the expression for $t(x)$ on the thermally insulated region, deduced from the approximate formula, is

$$t(x) = \frac{T_0}{\delta}\sqrt{1-x^2}\left[1 - \sum_{n=1}^{N-1}\sum_{i=1}^{n}\varepsilon_{ni}\delta^{-2n}x^{2i-2} + O(\delta^{-2N})\right]. \tag{17}$$

In this expression

$$\varepsilon_{11} = \frac{b_1}{2}, \quad \varepsilon_{22} = \frac{b_2}{2}, \quad \varepsilon_{33} = \frac{b_3}{2}, \quad \varepsilon_{21} = -\frac{b_1^2}{2},$$

$$\varepsilon_{31} = \frac{1}{4}b_1^3 - \frac{11}{16}b_1b_2 - \frac{11}{8}b_3, \quad \varepsilon_{32} = -\frac{1}{4}b_1b_2 + 4b_3.$$

The coefficients b_k calculated from Eq. (11) for $h = \infty$ are

$$b_1 = 0.8225, \quad b_2 = -0.1353, \quad b_3 = 0.0318.$$

The precise value of $t(x)$ found from Eq. (13) can be determined from

$$t(x) = \frac{T_0}{\pi}\sum_{k=1}^{2}\arcsin\left\{\operatorname{cth}\frac{\pi}{\delta} - \operatorname{csh}\frac{\pi}{\delta}\exp\left[(-1)^k\frac{\pi x}{\delta}\right]\right\}. \tag{18}$$

Table 1 gives the maximum temperatures t_e and t_a calculated from both the exact formula (18) and the approximate formula (17) for $x = 0$ for different values of $\delta = d/l$. It is clear from Table 1 that for $\delta = 1.5$ the approximate solution differs from the exact solution by less than 4%, and the relative error decreases rapidly with increasing δ.

2. Consider an infinite layer of thickness d on the upper surface of which we have conditions of the first, second, or third kind, and on the lower surface of which we have a given heat flux $q(r)$ on a circle of radius l, the remainder of this surface being kept at zero temperature. We shall solve the problem for the axially symmetric case. In terms of the dimensionless variables referred to the radius l, the boundary conditions are then of the form given by Eqs. (1) and (2) if we replace y by z and x by r in these conditions. The main temperature field will be assumed to be known as in the case of the band. Applying the integral Hankel transformation to the heat transfer equation for the additional temperature, and taking into account the corresponding boundary conditions, we have the following integral equation for the function $t(r) = t(r, 0)$:

$$\int_0^1 \rho t(\rho)\,d\rho \int_0^\infty \eta^2 L(\eta) J_0\left(\frac{\eta\rho}{\delta}\right) J_0\left(\frac{\eta r}{\delta}\right) d\eta = \delta^3 f(r) \quad (0 \leqslant r \leqslant 1), \tag{19}$$

where $f(r) = q(r) - \partial t^*(r, 0)/\partial z$, and the function $L(\eta)$ for the boundary conditions of the first, second, or third kind is given by Eqs. (7) and (6), respectively.

Equation (19) can be reduced to the Fredholm equation of the second kind [3]

$$t(r) = \frac{2}{\pi}\int_r^1 \frac{du}{\sqrt{u^2 - r^2}} \int_0^u \frac{\xi d\xi}{\sqrt{u^2 - \xi^2}} \left[f(\xi) - \frac{\pi}{2\delta^3}\int_0^1 \rho t(\rho) M\left(\frac{\xi}{\delta}, \frac{\rho}{\delta}\right) d\rho \right] \tag{20}$$

where

$$M\left(\frac{\xi}{\delta}, \frac{\rho}{\delta}\right) = \int_0^\infty \eta^2 [L(\eta) - 1] J_0\left(\frac{\xi\eta}{\delta}\right) J_0\left(\frac{\rho\eta}{\delta}\right) d\eta = \sum_{k=0}^\infty d_k \left(\frac{\rho}{\delta}\right)^{2k} F\left(-k, -k, 1, \frac{\xi^2}{\rho^2}\right),$$

$$d_k = \frac{(-1)^k}{2^{2k-1}\pi (k!)^2} \int_0^\infty [L(\eta) - 1]\,\eta^{2k+2} d\eta. \tag{21}$$

Substituting Eq. (7) in Eq. (21), we have, respectively, for the first and second boundary conditions

$$d_k^{(1)} = \frac{2(-1)^k (2k+2)!}{4^{2k+1}\pi (k!)^2} \zeta(2k+3), \quad d_k^{(2)} = [4^{-(k+1)} - 1]\, d_k^{(1)}. \tag{22}$$

For the third boundary condition the coefficients d_k are evaluated by numerical integration. The solution of the integral equation (20) will be sought in the form

$$t(r) = \sum_{n=0}^\infty \delta^{-n} t_n(r). \tag{23}$$

Substituting Eq. (23) in Eq. (20), and equating coefficients of equal powers of δ, we obtain a number of recurrence relations from which we successively determine the $t_n(r)$.

Consider a special case. Suppose that $f(r) = -T_0/\delta$ in Eq. (19). This corresponds to the case where on the upper surface of the layer ($z = \delta$) we specify the temperature $T = T_0$, and the circular part of the lower surface ($z = 0$, $r < 1$) is thermally insulated. The temperature $t(r)$ on this part of the lower surface, found by the above method, is

$$t(r) = \frac{2T_0}{\pi\delta}\sqrt{1 - r^2}\left[1 - \frac{d_0}{3\delta^3} - \left(\frac{7}{5} + r^2\right)\frac{4d_1}{27\delta^5} + \frac{d_0^2}{9\delta^6} - \left(\frac{17}{7} + 3r^2 + r^4\right)\frac{128 d_2}{675\delta^7} + \left(\frac{16}{5} + r^2\right)\frac{4 d_0 d_1}{81\delta^8} + O(\delta^{-9})\right]. \tag{24}$$

In this case,

$$d_0 = 0.3825, \quad d_1 = 0.2475, \quad d_2 = 0.1128.$$

The table gives the maximum value of the temperature, t_c, calculated from Eq. (24) for $r = 0$ for a number of values of $\delta = d/l$.

We note that, as the relative width of the band increases, the perturbed temperature field rapidly tends to the value obtained for the half-plane (this will be practically so for $d/l > 4$). The temperature $t(r)$ for the layer is practically the same as the temperature for the half-space for a still lower relative thickness of the layer. These results are of interest because they enable us to judge the validity of the expressions for a half-plane and half-space in the case of a band and a layer.

Having found the temperature t on that part of the lower surface of the band or layer on which the heat flux is specified, we know the limiting value of the temperature over the entire lower surface. The temperature inside the above regions can be determined through the corresponding Fourier or Hankel integral transformation.

We note in conclusion that the above results can also be used when $q = 0$ to determine the temperature field in a band $|y| \leq d$ and a layer $|z| \leq d$ in the presence of a thermally insulating crack lying along the middle of the band ($y = 0$) or on the middle plane of the layer ($z = 0$). In this case, the main temperature t^*, which is established in these regions in the absence of the crack, must be divided into a symmetric and an antisymmetric part relative to the line or plane containing the crack. The presence of the crack does not affect the symmetric part of the temperature field, whilst the antisymmetric part is perturbed and this perturbation can be determined as indicated above.

NOTATION

T	is the required temperature;
t^*	is the temperature for homogeneous boundary condition on the lower face;
t	is the additional temperature due to the mixed boundary conditions;
T_c	is the temperature of ambient medium;
q	is the heat flux specified on the lower face of the band or layer;
x, y, z	are the Cartesian coordinates;
$\delta = d/l$;	
d	is the width of band or thickness of layer;
l	is the half length of region or radius of the circle on the lower surface of band or layer, respectively, on which the heat flux is specified;
h	is the heat-transfer coefficient;
J_0	is the Bessel function;
F	is the hypergeometric function;
ζ	is the zeta function;
B_{2k}	are the Bernouilli numbers.

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